Indian Statistical Institute, Bangalore
M. Math First Year
Second Semester - Complex Analysis

Semestral Exam — Date: May 07, 2018
Maximum marks: 100 — Duration: 3 hours

1. Let f be a holomorphic function from the unit disc $\mathbb{D}$ into itself. Then show that, for any $z \in \mathbb{D}, |f'(z)| \leq \frac{1-|f(z)|^2}{1-|z|^2}$. Find all f for which equality holds here for some z. (Hint: Use Mobius maps to reduce it to Schwarz Lemma.) [15 + 5 = 20]

2. Let $\Omega = \{z = x + iy : x \in \mathbb{R}, y \in (-\frac{\pi}{2}, \frac{\pi}{2})\}$.

   (a) Give an example of a holomorphic function f on Ω such that f is continuous on $\overline{\Omega}$, bounded on $\partial\Omega$, but unbounded on Ω.

   (b) Suppose f is holomorphic Ω, continuous on $\overline{\Omega}$ and, further, there is a constant $\alpha < 1$ such that $|f(z)| < exp\ (exp(\alpha|x|))$ for all $z = x + iy \in \Omega$. If f is bounded on $\partial\Omega$ then show that f must be bounded on Ω. (Hint: For part(b), take $\epsilon\ > 0$ and $\beta \in (\alpha, 1)$. Look at the function $z \to exp(-\epsilon\ (e^{\beta z} + e^{-\beta z}))$. [5 + 15 = 20]

3. State and prove Cauchy's integral formula for general domains. [5 + 15 = 20]

4. Let $H = \{z = x + iy : x \in \mathbb{R}, y > 0\}$. Define $G : H \to \mathbb{C}$ by $G(z) = \sum\limits_{m,\ n \in \mathbb{Z}\setminus(0,0)} (m + nz)^{-4}$.

   (a) Show that G is holomorphic on H.

   (b) If $g \in GL(2, Z)$ and $z \in H$, find $G(gz)$ in terms of $G(z)$. (Here, for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, gz = \frac{az+b}{cz+d}$.) [10 + 10 = 20]

5. Let Ω be a planar domain, $\{f_n\}$ a sequence of holomorphic functions on Ω and f a holomorphic function on Ω such that $f_n \to f$ locally uniformly on Ω as $n \to \infty$. Show that

   (a) $f_n' \to f'$ locally uniformly on Ω.

   (b) If each f_n is non-vanishing on Ω then either $f \equiv 0$ or f is non-vanishing on Ω,

   (c) If each f_n is one-one on Ω, then either f is constant or f is one-one on Ω.

   (d) Give examples to show that both cases of (b) and (c) can actually occur. [5 + 5 + 5 + 5 = 20]